Kuwait University | Calculus II : Math 102 | 14 January 2012
Department of Mathematics | Final Examination | Duration : 120 minutes

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The use of a calculator of any kind is not allowed. All communication devices including mobile telephones should be switched off. Answer all of the following questions.

1. Prove the following. **(3+1 pts)**

   (a) $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$ if $xy < 1$

   (b) $\arctan(1/2) + \arctan(1/3) = \pi/4$

2. Find $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$. **(4 pts)**

3. Sketch the parametric curve $x = 1+t$ and $y = t^2 - 4t$ for $0 \leqslant t \leqslant 5$. Indicate with an arrow the direction in which the curve is traced as t increases. **(4 pts)**

4. Evaluate $\displaystyle\int \frac{dx}{1+\sin x - \cos x}$. **(4 pts)**

5. Evaluate $\displaystyle\int_0^{\pi} \frac{\sec^2 x}{4+\tan^2 x}\,dx$. **(4 pts)**

6. Find the area of the surface obtained by rotating the circle $x^2 + (y-1)^2 = 1$ about the line $y = 0$. **(5 pts)**

7. Find the arc length of the curve given by $x = 1 + e^{2t}$ and $y = 1 - e^t$ for $-\ln 2 \leqslant t \leqslant 0$. **(5 pts)**

8. Find the area of the region that lies inside the polar curve $r = 1 - \sin\theta$ and outside the polar curve $r = 1$. **(5 pts)**

9. Find the centre of mass of the region above the x-axis and below the curve $x = t - \sin t$ and $y = 1 - \cos t$ for $0 \leqslant t \leqslant 2\pi$. **(5 pts)**

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SOLUTION

1. (a) METHOD 1. Use

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

with $a = \arctan x$ and $b = \arctan y$. This implies

$$\tan(\arctan x + \arctan y) = \frac{x+y}{1-xy}$$

if $xy \neq 1$. Hence,

$$\arctan\big(\tan(\arctan x + \arctan y)\big) = \arctan\left(\frac{x+y}{1-xy}\right).$$

The desired statement follows given that

$$\pi/2 < \arctan x + \arctan y < \pi/2 \quad \text{if } xy < 1.$$

To show that this inequality is true, consider four cases. First, $x \leqslant 0 \leqslant y$. In this case, $-\pi/2 < \arctan x \leqslant 0$ and $0 \leqslant \arctan y < \pi/2$. Second, $y \leqslant 0 \leqslant x$. This is similar. Third, $y > 0$ and $0 < x < 1/y$. In this case, $0 < \arctan x + \arctan y < \arctan(1/y) + \arctan y = \pi/2$. Fourth, $y < 0$ and $1/y < x < 0$. In this case, $0 > \arctan x + \arctan y = -\arctan|x| - \arctan|y| > -\pi/2$ by the previous case.

METHOD 2. View y as an arbitrary constant. Then for all x such that $xy < 1$,

$$\begin{aligned}\frac{d}{dx}\arctan\left(\frac{x+y}{1-xy}\right) &= \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\,\frac{(1)(1-xy)-(x+y)(-y)}{(1-xy)^2} = \cdots \\ &= \frac{1}{1+x^2}.\end{aligned}$$

Hence, for all such x,

$$\arctan\left(\frac{x+y}{1-xy}\right) = \arctan x + C,$$

where C is a constant. Substituting $x = 0$ gives

$$C = \arctan y.$$

(b) Substituting $x = 1/2$ and $y = 1/3$ in the statement of part (a) yields

$$\arctan(1/2) + \arctan(1/3) = \arctan\frac{(1/2)+(1/3)}{1-(1/2)(1/3)} = \cdots = \arctan 1 = \frac{\pi}{4}.$$

2. Take the logarithm.

$$\lim_{x\to 1}\ln\Big[(2-x)^{\tan(\pi x/2)}\Big] = \lim_{x\to 1}\tan(\pi x/2)\ln(2-x) = \lim_{x\to 1}\frac{\ln(2-x)}{\cot(\pi x/2)} = \frac{0}{0},$$

which is indeterminate.

$$\lim_{x\to 1}\frac{\frac{d}{dx}\ln(2-x)}{\frac{d}{dx}\cot(\pi x/2)} = \lim_{x\to 1}\frac{-1/(2-x)}{[-\csc^2(\pi x/2)](\pi/2)} = \cdots = \frac{2}{\pi}.$$

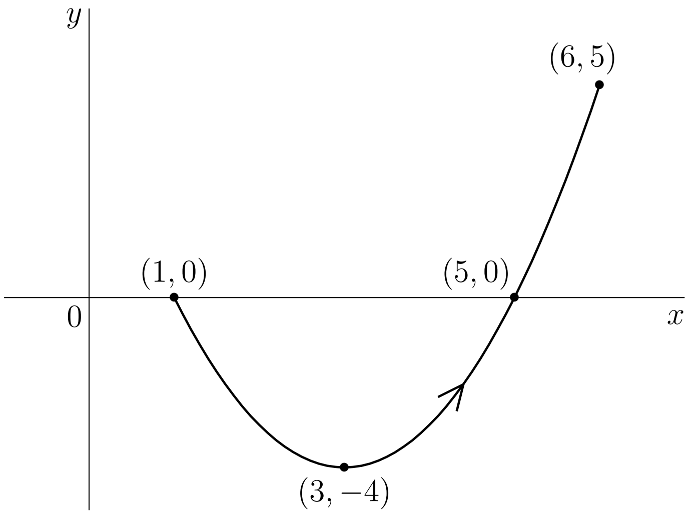
So, by L'Hospital's Rule,

$$\lim_{x\to 1}\ln\Big[(2-x)^{\tan(\pi x/2)}\Big] = 2/\pi.$$

Hence,

$$\lim_{x\to 1}(2-x)^{\tan(\pi x/2)} = e^{2/\pi}.$$

3. Eliminate $t = x - 1$. Then $y = (x-1)^2 - 4(x-1) = (x-1)(x-5) = (x-3)^2 - 4$ for $1 \leqslant x \leqslant 6$. This describes a segment of the parabola with vertex $(3, -4)$ and x-intercepts at 1 and 5.



4. Use the Method of Weierstrass, i.e. substitute $t = \tan(x/2)$. This gives

$$\begin{aligned}\int\frac{dx}{1+\sin x-\cos x} &= \int\frac{1}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \cdots = \int\frac{1}{t(1+t)}\,dt\\ &= \int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt = \ln|t| - \ln|1+t| + C\\ &= -\ln|1+\cot(x/2)| + C.\end{aligned}$$

5. By symmetry,[1]

$$\int_0^{\pi} \frac{\sec^2 x}{4+\tan^2 x}\,dx = 2\int_0^{\pi/2} \frac{\sec^2 x}{4+\tan^2 x}\,dx = \arctan\left(\frac{\tan x}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{2}.$$

6. Parameterize the circle with $x = \cos t$ and $y = 1+\sin t$ for $0 \leqslant t \leqslant 2\pi$. Then the surface area is

$$\begin{aligned} S &= \int_?^? 2\pi y\,ds = \int_0^{2\pi} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \cdots = \int_0^{2\pi} 2\pi y\,dt \\ &= 2\pi\int_0^{2\pi} (1+\sin t)\;dt = 2\pi\,(t-\cos t)\Big|_0^{2\pi} = 4\pi^2. \end{aligned}$$

7. The arc length

$$L = \int_{-\ln 2}^0 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_{-\ln 2}^0 \sqrt{4e^{4t}+e^{2t}}\,dt = \int_{-\ln 2}^0 e^t\sqrt{4e^{2t}+1}\,dt.$$

Substitute $2e^t = u$ and hence $e^t\,dt = (1/2)\,du$, then $u = \tan\theta$ and hence $du = \sec^2\theta\,d\theta$, and finally $v = \sin\theta$ and hence $dv = \cos\theta\,d\theta$ to obtain

$$\begin{aligned} L &= \frac{1}{2}\int_1^2 \sqrt{u^2+1}\,du = \frac{1}{2}\int_{\pi/4}^{\arctan 2} \sqrt{\tan^2\theta+1}\sec^2\theta\,d\theta \\ &= \frac{1}{2}\int_{\pi/4}^{\arctan 2} \sec^3\theta\,d\theta = \frac{1}{2}\int_{\pi/4}^{\arctan 2} \frac{\cos\theta}{(1-\sin^2\theta)^2}\,d\theta \\ &= \frac{1}{2}\int_{1/\sqrt{2}}^{2/\sqrt{5}} \frac{1}{(1-v^2)^2}\,dv. \end{aligned}$$

The partial fractions of the integrand are

$$\frac{A}{v-1} + \frac{B}{(v-1)^2} + \frac{C}{v+1} + \frac{D}{(v+1)^2},$$

where

$$\begin{aligned} 1 &= A(v-1)(v+1)^2 + B(v+1)^2 + C(v-1)^2(v+1) + D(v-1)^2 \\ &= A(v^3+v^2-v-1) + B(v^2+2v+1) \\ &\quad + C(v^3-v^2-v+1) + D(v^2-2v+1). \end{aligned}$$

[1] Although the integrand has a discontinuity at $\pi/2$, this is removable. So the integral is proper.

$$\int_0^{\pi} \frac{\sec^2 x}{4+\tan^2 x}\,dx \neq \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)\Big|_0^{\pi}$$

because $\frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)$ is not a continuous antiderivative of $\frac{\sec^2 x}{4+\tan^2 x}$ in $[0,\pi]$.

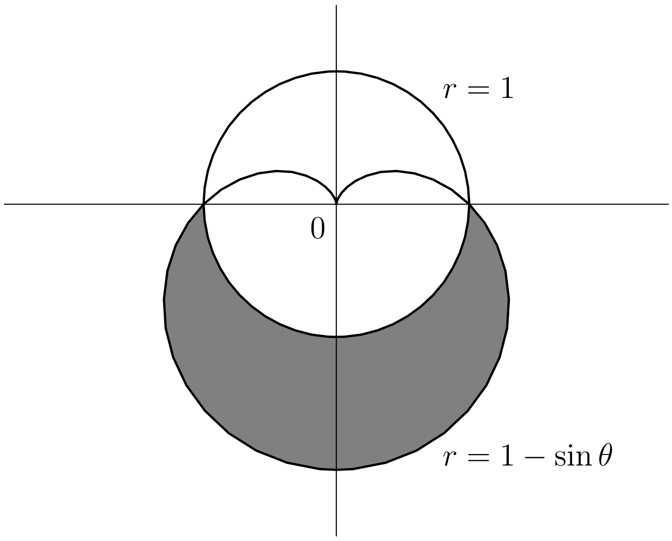
Equating coefficients yields

$$\begin{cases} A+C = 0 \\ A+B-C+D = 0 \\ -A+2B-C-2D = 0 \\ -A+B+C+D = 1 \end{cases} \Rightarrow \quad \cdots \quad \Rightarrow \begin{cases} A = -1/4 \\ B = 1/4 \\ C = 1/4 \\ D = 1/4. \end{cases}$$

Hence,

$$\begin{aligned} L &= \frac{1}{8}\int_{1/\sqrt{2}}^{2/\sqrt{5}} \left[-\frac{1}{v-1} + \frac{1}{(v-1)^2} + \frac{1}{v+1} + \frac{1}{(v+1)^2}\right] dv \\ &= \frac{1}{8}\left(-\ln|v-1| - \frac{1}{v-1} + \ln|v+1| - \frac{1}{v+1}\right)\Bigg|_{1/\sqrt{2}}^{2/\sqrt{5}} = \cdots \\ &= \frac{1}{4}\left[2\sqrt{5} - \sqrt{2} + \ln(2+\sqrt{5}) - \ln(1+\sqrt{2})\right]. \end{aligned}$$

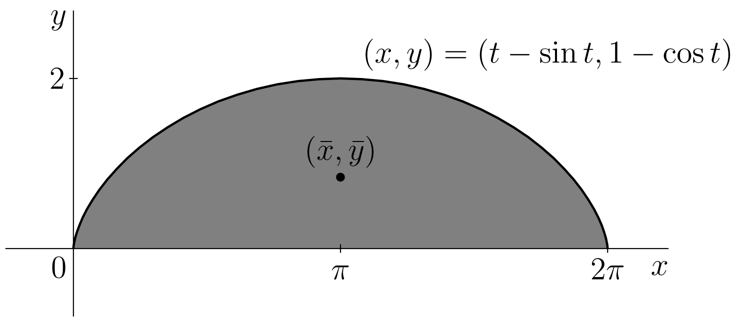
8. The first curve is a cardioid, the second is a circle.



The curves intersect when $r = 1 - \sin\theta = 1 \;\Rightarrow\; \sin\theta = 0 \;\Rightarrow\; \theta = k\pi$ for integer k. Thus the area is

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi}^{0} \left[(1-\sin\theta)^2 - 1^2\right] d\theta = \frac{1}{2}\int_{-\pi}^{0} \left(\sin^2\theta - 2\sin\theta\right) d\theta \\ &= \frac{1}{4}\int_{-\pi}^{0} (1 - \cos 2\theta - 4\sin\theta)\, d\theta = \frac{1}{4}\left(\theta - \frac{1}{2}\sin 2\theta + 4\cos\theta\right)\Bigg|_{-\pi}^{0} = \frac{\pi+8}{4}. \end{aligned}$$

9. The curve is a cycloid.



By symmetry the x-coordinate of the centre of mass is $\bar{x} = \pi$. The y-coordinate of the centre of mass is

$$\bar{y} = \frac{1}{2A} \int_0^{2\pi} y^2\, dx,$$

where A is the area, given by

$$\begin{aligned} A &= \int_0^{2\pi} y\, dx = \int_0^{2\pi} y \frac{dx}{dt}\, dt = \int_0^{2\pi} (1 - \cos t)(1 - \cos t)\, dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\, dt = \frac{1}{2} \int_0^{2\pi} (3 - 4\cos t + \cos 2t)\, dt \\ &= \frac{1}{2} \left(3t - 4\sin t - \frac{1}{2} \sin 2t \right) \Big|_0^{2\pi} = 3\pi. \end{aligned}$$

Hence,

$$\begin{aligned} \bar{y} &= \frac{1}{6\pi} \int_0^{2\pi} y^2\, dx = \frac{1}{6\pi} \int_0^{2\pi} y^2 \frac{dx}{dt}\, dt = \frac{1}{6\pi} \int_0^{2\pi} (1 - \cos t)^3\, dt \\ &= \frac{1}{6\pi} \int_0^{2\pi} (1 - 3\cos t + 3\cos^2 t - \cos^3 t)\, dt = \cdots \\ &= \frac{1}{12\pi} \int_0^{2\pi} (5 - 8\cos t + 3\cos 2t + 2\sin^2 t \cos t)\, dt \\ &= \frac{1}{12\pi} \left(5t - 8\sin t + \frac{3}{2} \sin 2t + \frac{2}{3} \sin^3 t \right) \Big|_0^{2\pi} = \cdots = \frac{5}{6}. \end{aligned}$$

Answer: $(\pi, 5/6)$.